３年普通科　数学 III　　定積分演習プリント１

1　次の定積分を求めよ。

(1) $\displaystyle\int_1^4 \frac{1+x}{\sqrt{x}}dx$

(2) $\displaystyle\int_0^{\frac{\pi}{3}} \tan\theta d\theta$

(3) $\displaystyle\int_{-\frac{\pi}{6}}^{\pi} \cos^2 2\theta d\theta$

(4) $\displaystyle\int_0^2 \frac{x}{(3-x)^2}dx$

(5) $\displaystyle\int_0^1 xe^{-x^2}dx$

(6) $\displaystyle\int_1^4 \sqrt{x}\log x dx$

３年普通科　数学 III　　定積分演習プリント１　解答例

1　次の定積分を求めよ。

(1) $\displaystyle\int_1^4 \frac{1+x}{\sqrt{x}}dx$

$$(\text{与式}) = \int_1^4 \left(\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}}\right)dx = \int_1^4 \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)dx = \int_1^4 (x^{-\frac{1}{2}} + x^{\frac{1}{2}})dx$$

$$= \left[\frac{2}{1}x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \left[2\sqrt{x} + \frac{2}{3}x\sqrt{x}\right]_1^4$$

$$= \left(2\sqrt{4} + \frac{2}{3}\cdot 4\sqrt{4}\right) - \left(2\sqrt{1} + \frac{2}{3}\cdot 1\sqrt{1}\right)$$

$$= 2(\sqrt{4} - \sqrt{1}) + \frac{2}{3}(4\sqrt{4} - 1\sqrt{1}) = 2(2-1) + \frac{2}{3}(8-1)$$

$$= 2\cdot 1 + \frac{2}{3}\cdot 7 = \frac{20}{3} \quad //$$

(2) $\displaystyle\int_0^{\frac{\pi}{3}} \tan\theta d\theta$

$$(\text{与式}) = \int_0^{\frac{\pi}{3}} \frac{\sin\theta}{\cos\theta}\,d\theta = -\int_0^{\frac{\pi}{3}} \frac{(\cos\theta)'}{\cos\theta}\,d\theta = -\Big[\log(\cos\theta)\Big]_0^{\frac{\pi}{3}}$$

$$= -\left\{\log\left(\cos\frac{\pi}{3}\right) - \log\left(\cos 0\right)\right\}$$

$$= -\left(\log\frac{1}{2} - \log 1\right) = -\log\frac{1}{2} = \log\left(\frac{1}{2}\right)^{-1} = \log 2 \quad //$$

(3) $\displaystyle\int_{-\frac{\pi}{6}}^{\pi} \cos^2 2\theta d\theta$

$\cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta)$ より，$\cos^2 2\theta = \dfrac{1}{2}(1+\cos 4\theta)$ だから

$$(\text{与式}) = \int_{-\frac{\pi}{6}}^{\pi} \frac{1}{2}(1+\cos 4\theta)d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{6}}^{\pi}$$

$$= \frac{1}{2}\left\{\left(\pi - \frac{1}{4}\sin 4\pi\right) - \left(-\frac{\pi}{6} - \frac{1}{4}\sin\left(-\frac{2}{3}\pi\right)\right)\right\}$$

$$= \frac{1}{2}\left\{\left(\pi - \frac{1}{4}\cdot 0\right) - \left(-\frac{\pi}{6} - \frac{1}{4}\cdot\left(-\frac{\sqrt{3}}{2}\right)\right)\right\}$$

$$= \frac{1}{2}\left(\pi + \frac{\pi}{6} + \frac{\sqrt{3}}{8}\right) = \frac{7}{12}\pi + \frac{\sqrt{3}}{16} \quad //$$

(4) $\displaystyle\int_0^2 \frac{x}{(3-x)^2}dx$

$3 - x = t$ とおくと

$x = -t + 3,$

$dx = (-1)dt$

x	$0 \to 2$
t	$3 \to 1$

$$(\text{与式}) = \int_3^1 \frac{-t+3}{t^2}\cdot(-1)dt = \int_3^1 \frac{t-3}{t^2}dt = \int_3^1 \left(\frac{1}{t} - 3t^{-2}\right)dt$$

$$= \left[\log t - 3\cdot\frac{1}{-1}t^{-1}\right]_3^1$$

$$= \left[\log t + \frac{3}{t}\right]_3^1 = \left(\log 1 + \frac{3}{1}\right) - \left(\log 3 + \frac{3}{3}\right)$$

$$= (0+3) - (\log 3 + 1) = 2 - \log 3 \quad //$$

(5) $\displaystyle\int_0^1 xe^{-x^2}dx$

$-x^2 = t$ とおくと

$-2x\,dx = dt$

x	$0 \to 1$
t	$0 \to -1$

$$(\text{与式}) = -\frac{1}{2}\int_0^1 e^{-x^2}\cdot(-2x)dx = -\frac{1}{2}\int_0^{-1} e^t\,dt = -\frac{1}{2}\Big[e^t\Big]_0^{-1}$$

$$= -\frac{1}{2}\left(e^{-1} - e^0\right) = \frac{1}{2}\left(1 - \frac{1}{e}\right) \quad //$$

[別解]

$$(\text{与式}) = -\frac{1}{2}\int_0^1 e^{-x^2}\cdot(-2x)dx = -\frac{1}{2}\int_0^1 e^{-x^2}\cdot(-x^2)'dx$$

$$= -\frac{1}{2}\Big[e^{-x^2}\Big]_0^1$$

$$= -\frac{1}{2}\left(e^{-1} - e^0\right) = \frac{1}{2}\left(1 - \frac{1}{e}\right) \quad //$$

(6) $\displaystyle\int_1^4 \sqrt{x}\log x dx$

$\displaystyle\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C$　　(C は積分定数)　であるから

$$(\text{与式}) = \left[\frac{2}{3}x^{\frac{3}{2}}\cdot\log x\right]_1^4 - \int_1^4 \frac{2}{3}x^{\frac{3}{2}}\cdot\frac{1}{x}\,dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\cdot\log x\right]_1^4 - \frac{2}{3}\int_1^4 x^{\frac{1}{2}}\,dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\cdot\log x\right]_1^4 - \frac{2}{3}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$$

$$= \frac{2}{3}\left[x\sqrt{x}\cdot\log x\right]_1^4 - \frac{4}{9}\left[x\sqrt{x}\right]_1^4$$

$$= \frac{2}{3}\left(4\sqrt{4}\log 4 - 1\sqrt{1}\log 1\right) - \frac{4}{9}\left(4\sqrt{4} - 1\sqrt{1}\right)$$

$$= \frac{2}{3}\left(8\cdot 2\log 2 - 0\right) - \frac{4}{9}\left(8 - 1\right)$$

$$= \frac{2}{3}\cdot 16\log 2 - \frac{4}{9}\cdot 7$$

$$= \frac{32}{3}\log 2 - \frac{28}{9} \quad //$$

３年普通科　数学 III　　定積分演習プリント２

2　a を定数とする。次の定積分を求めよ。

(1) $\displaystyle\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx$

(2) $\displaystyle\int_{-a}^{a} \frac{x^2}{x^2+a^2}\,dx$

3　次の関数を x で微分せよ。ただし，(2) では $x>0$ とする。

(1) $\displaystyle\int_{x}^{2x} \sin\theta\,d\theta$

(2) $\displaystyle\int_{x}^{x^2} \log t\,dt$

4　等式 $f(x)=x+\displaystyle\int_0^{\pi} f(t)\sin t dt$ を満たす関数 $f(x)$ を求めよ。

３年普通科　数学 III　　定積分演習プリント２　解答例

2　a を定数とする。次の定積分を求めよ。

(1) $\displaystyle\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx$

偶関数なので

$$(\text{与式}) = 2\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

x	$0 \to a$
θ	$0 \to \frac{\pi}{2}$

$x = a\sin\theta$ とおくと

$dx = a\cos\theta\,d\theta$

$\sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta)$ より，$\sin^2 2\theta = \dfrac{1}{2}(1-\cos 4\theta)$ だから

$$\begin{aligned}
(\text{与式}) &= 2\int_0^{\frac{\pi}{2}} (a\sin\theta)^2\sqrt{a^2-(a\sin\theta)^2}\,a\cos\theta\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} a^2\sin^2\theta\cdot a\cos\theta\ \cdot a\cos\theta\,d\theta \\
&= 2a^4\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta = \frac{a^4}{2}\int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2\,d\theta \\
&= \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\,d\theta = \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 4\theta)\,d\theta \\
&= \frac{a^4}{4}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\,d\theta = \frac{a^4}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}} \\
&= \frac{a^4}{4}\left\{\left(\frac{\pi}{2}-\frac{1}{4}\sin 2\pi\right)-\left(0-\frac{1}{4}\sin 0\right)\right\} \\
&= \frac{a^4}{4}\cdot\frac{\pi}{2} = \frac{\pi a^4}{8}\ /\!/
\end{aligned}$$

(2) $\displaystyle\int_{-a}^{a} \frac{x^2}{x^2+a^2}\,dx$

偶関数なので $(\text{与式}) = 2\displaystyle\int_0^a \frac{x^2}{x^2+a^2}\,dx$

x	$0 \to a$
θ	$0 \to \frac{\pi}{4}$

$x = a\tan\theta$ とおくと

$dx = \dfrac{a}{\cos^2\theta}d\theta$

$1+\tan^2\theta = \dfrac{1}{\cos^2\theta}$ より

$\leftarrow \quad \dfrac{1}{1+\tan^2\theta} = \cos^2\theta$

$\leftarrow \quad \tan^2\theta = \dfrac{1}{\cos^2\theta} - 1$

$$\begin{aligned}
(\text{与式}) &= 2\int_0^{\frac{\pi}{4}} \frac{(a\tan\theta)^2}{(a\tan\theta)^2+a^2}\cdot\frac{a}{\cos^2\theta}d\theta \\
&= 2\int_0^{\frac{\pi}{4}} \frac{a^2\tan^2\theta}{a^2(\tan^2\theta+1)}\cdot\frac{a}{\cos^2\theta}d\theta \\
&= 2\int_0^{\frac{\pi}{4}} \frac{a^2\tan^2\theta}{a^2}\cdot\cos^2\theta\cdot\frac{a}{\cos^2\theta}d\theta \\
&= 2a\int_0^{\frac{\pi}{4}} \tan^2\theta\,d\theta \\
&= 2a\int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2\theta}-1\right)d\theta \\
&= 2a\Big[\tan\theta-\theta\Big]_0^{\frac{\pi}{4}} \\
&= 2a\left\{\left(\tan\frac{\pi}{4}-\frac{\pi}{4}\right)-\left(\tan 0-0\right)\right\} \\
&= 2a\left(1-\frac{\pi}{4}\right)\ /\!/
\end{aligned}$$

3　次の関数を x で微分せよ。ただし，(2) では $x>0$ とする。

(1) $\displaystyle\int_x^{2x} \sin\theta\,d\theta$

$$\int_x^{2x} \sin\theta\,d\theta = \Big[-\cos\theta\Big]_x^{2x} = (-\cos 2x)-(-\cos x) \quad \text{であるから}$$

$$\begin{aligned}
\frac{d}{dx}\int_x^{2x} \sin\theta\,d\theta &= \frac{d}{dx}\Big\{(-\cos 2x)-(-\cos x)\Big\} \\
&= \sin 2x\cdot 2-\sin x \\
&= 2\sin 2x-\sin x\ /\!/
\end{aligned}$$

(2) $\displaystyle\int_x^{x^2} \log t\,dt$

$$\int_x^{x^2} \log t\,dt = \Big[t\log t - t\Big]_x^{x^2} = (x^2\log x^2 - x^2)-(x\log x - x)$$

$$\begin{aligned}
&= 2x^2\log x - x^2 - x\log x + x \\
&= (2x^2-x)\log x - x^2 + x \quad \text{であるから}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\int_x^{x^2} \log t\,dt &= \frac{d}{dx}\Big\{(2x^2-x)\log x - x^2 + x\Big\} \\
&= (4x-1)\log x + (2x^2-x)\cdot\frac{1}{x} - 2x + 1 \\
&= (4x-1)\log x + (2x-1) - 2x + 1 \\
&= (4x-1)\log x\ /\!/
\end{aligned}$$

4　等式 $f(x) = x + \displaystyle\int_0^{\pi} f(t)\sin t dt$ を満たす関数 $f(x)$ を求めよ。

$\displaystyle\int_0^{\pi} f(t)\sin t dt = a$ とおくと $f(x) = x + a$

$$\begin{aligned}
\int_0^{\pi} f(t)\sin t dt &= \int_0^{\pi} (t+a)\sin t dt \\
&= \Big[(t+a)\cdot(-\cos t)\Big]_0^{\pi} - \int_0^{\pi} 1\cdot(-\cos t)dt \\
&= (\pi+a)\cdot(-\cos\pi)-(0+a)\cdot(-\cos 0) + \int_0^{\pi}\cos t dt \\
&= \pi + a + a + \Big[\sin t\Big]_0^{\pi} \\
&= \pi + 2a + \sin\pi - \sin 0 \\
&= \pi + 2a
\end{aligned}$$

よって，$a = \pi + 2a$ より $a = -\pi$

求める関数は，$f(x) = x - \pi$ $/\!/$